

TOWARD A CONJECTURE OF TAN AND TU ON FIBERED GENERAL TYPE SURFACES

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ABSTRACT. Given a semistable non-isotrivial fibered surface $f : X \rightarrow \mathbb{P}^1$ it was conjectured by Tan and Tu that if X is of general type, then f admits at least 7 singular fibers. In this paper we prove this conjecture in several particular cases, i.e. assuming f is obtained from blowing-up the base locus of a transversal pencil on an exceptional minimal surface S or assuming that f is obtained as the blow-up of the base locus of a transversal and adjoint pencil on a minimal surface.

1. NOTATION

We work on the complex field number \mathbb{C} . All the considered varieties will be assumed irreducible and projective. Through the paper we shall use the following notation:

- . X will be a general type surface and S its minimal model. $\pi : X \rightarrow S$ will be the associated chain of blowing-downs.
- . $f : X \rightarrow \mathbb{P}^1$ will be a semi-stable, non-isotrivial fibration, and F the general fibre. We set g = the genus of F . By C we will denote the image of F under π and

$$\Lambda : S \dashrightarrow \mathbb{P}^1,$$

the pencil induced by f . We denote by s the number of singular fibers.

- . We shall say that Λ is transversal if its general member $C \in \Lambda$ is non-singular and intersects transversally any other general member $C' \in \Lambda$.
- . We'll freely use the standard notation in surfaces' theory. In particular $q = h^1(X, \mathcal{O}_X)$ will be the irregularity of X and $p_g = h^0(X, K_X)$ its geometric genus. By $e(X)$ will be denote the topological Euler characteristic.
- . Given divisors D_1 and D_2 in an algebraic surface we denote as usual $D_1 \equiv D_2$ for the numeric equivalence and $D_1 \sim D_2$ for the linear one. Most of the time we will be working on regular surfaces and in this case we indistinctly use both symbols.
- . The number m will be:

$$m := K_S^2 - K_X^2 = e(X) - e(S).$$

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Note that in general $m \leq C^2$. Adjunction Formula gives the important inequality $C.K_S + m \leq 2(g-1)$ with equality holding if Λ is transversal.

2. INTRODUCTION

Let $f : X \rightarrow \mathbb{P}^1$ be a non-isotrivial semistable fibered surface. It is a classical result that such a fibration admits a certain number of singular fibers (in contrast to the case when the base of the fibration is not rational or elliptic [1]). In the seminal paper [2] it was proved that this number s must be at least 4. Subsequently the bound have been sharpened to $s \geq 5$ if $g \geq 2$ and $s \geq 6$ if the surface X is not birationally ruled ([11] [12], [13]). It was conjectured by Tan and Tu, in a preprint previous to [13] that this bound must raise to 7 if X is of general type (Tan-Tu conjecture for what follows). They also proved the conjecture for genus $2 \leq g \leq 4$ and characterized fibrations of genus 5 with $s = 6$ on a general type surface as those obtained from blowing up the base locus of a transversal pencil on a Horikawa surface. Using this characterization the proof of the conjecture for $g = 5$ was completed in [15].

Roughly speaking the proof of these bounds are based, in case $g \geq 2$ on the canonical class inequality:

$$K_f^2 < 2(g-1)(2g_B - 2 + s),$$

for any non iso-trivial semi-stable fibration $f : X \rightarrow B$ of genus $g \geq 2$. Here K_f is the relatively canonical divisor, $K_f := K_X - f^*K_B$, which turns out to be $K_X(-2F)$ if $B = \mathbb{P}^1$. The bounds for s are obtained from the positivity properties of K_f and $K_f(-F)$.

Unfortunately, this approach is useless for proving Tan-Tu conjecture, since in this case the only relevant information the inequality provides is that $K_X^2 < 0$ if $s = 6$. However for most of the cases a fibered surface (of general type or not) satisfies $K_X^2 < 0$. Indeed, such a surface is obtained by blowing up the base locus of some pencil Λ on a minimal surface S .

In this paper we deal with Tan-Tu conjecture in some particular cases. We first impose on the minimal model S of X the conditions of being exceptional in the sense that either $K_S^2 = 2, p_g = 3$ or $K_S^2 = 1$ and $p_g = 2$ (see [1], VII.8) and assuming that f is obtained as the blowing-up of the base locus of a transversal pencil Λ in S . In these cases we are able to prove the conjecture using explicit descriptions of these surfaces as double coverings of rational surfaces, by means of the canonical or bi-canonical map. This is the content of Section 3, Theorems 4.1 and 4.2. In this sense there is some hope of extending the result to a wider class of surfaces.

Next, in section 5 we prove the conjecture assuming that the pencil Λ is adjoint, i.e. $C = B + K_S$ with B a big and nef divisor in S and $K_S^2 \geq 3$ and for $K_S^2 \leq 2$ assuming not only that Λ is adjoint, but also transversal (Theorem 5.3). The case $K_S^2 = p_g = 1$ is the subtler and is stated and proved in Proposition 5.2.

We list below the cases in which Tan-Tu conjecture have been proved in this article:

- . If $\chi(\mathcal{O}_S) = 1$ with no extra assumption on S or Λ .
- . If either $K_S^2 = 1, p_g = 2$ or $K_S^2 = 2, p_g = 3$ assuming Λ is transversal.
- . If Λ is transversal and adjoint and $K_S^2 \leq 2$ or if Λ is merely adjoint and $K_S^2 \geq 3$.

3. SOME GENERAL FACTS AND RESULTS

The following inequality will be systematically used: given a semi-stable non-isotrivial fibration of genus $g \geq 2$, $f : X \rightarrow B$, for any integer $e \geq 2$:

$$\frac{1}{3}e^2(K_X^2 - 2(g-1)(6(g_B-1) + s - s/e)) \leq e_f.$$

Original formulation involves the number of (-2) vertical curves in X , but for our purposes this version will be sufficient. The proof can be founded in [12] and is based on successive changes of the base B of the fibration. We call this Tan's inequality. In particular if $B = \mathbb{P}^1$ (our interest's case) and $s = 6$ we obtain:

$$(1) \quad \frac{1}{3}e(K_X^2 e + 12(g-1)) \leq e_f.$$

Useful forms of this inequality are collected in the following:

Lemma 3.1. *Let $f : X \rightarrow \mathbb{P}^1$ be semistable, non-isotrivial of genus $g \geq 2$. If $s = 6$, then evaluating (1) we obtain:*

i)

$$K_S^2 + C.K_S \leq 3\chi(\mathcal{O}_S) \text{ if } e = 3,$$

ii)

$$19K_S^2 + 18C.K_S \leq m + 36\chi(\mathcal{O}_S) \text{ if } e = 4,$$

iii)

$$7K_S^2 + 6C.K_S \leq m + 9\chi(\mathcal{O}_S) \text{ if } e = 5.$$

Proof. Evaluate (1) at the indicated value of e and substitute:

$$m + C.K_S \leq 2(g-1),$$

$$e(X) = 12\chi(\mathcal{O}_X) - K_X^2 \text{ (Noether's Formula),}$$

$$K_X^2 = K_S^2 - m,$$

and

$$e_f = 4(g-1) + e(X) \text{ (because } f \text{ is semistable) .}$$

□

We start by sharpening the bound for m obtained in [13], inequality (3.2) (compare with the proof of Theorem 2.1(4) in [13]).

Lemma 3.2. *Assume $f : X \rightarrow \mathbb{P}^1$ is a semistable fibration and the minimal model S of X is a general type surface, then:*

$$m \leq C^2 \leq \frac{4(g-1) + K_S^2 - \sqrt{8(g-1)K_S^2 + (K_S^2)^2}}{2}.$$

Proof. From Index Hodge Theorem applied to K_S and C we get:

$$mK_S^2 \leq C^2K_S^2 \leq (C.K_S)^2.$$

Adjoint formula gives $C.K_S \leq 2(g-1) - m$, therefore

$$0 \leq m^2 - (4(g-1) + K_S^2)m + 4(g-1)^2.$$

Consider the right hand term of the previous inequality as a polynomial in m . Its discriminant turns out to be

$$\Delta = 8(g-1)K_S^2 + (K_S^2)^2.$$

Thus its roots are:

$$m_{\pm} = \frac{4(g-1) + K_S^2 \pm \sqrt{\Delta}}{2}.$$

It follows that either $m \leq m_-$ or $m \geq m_+$. We claim that $m \leq m_-$ is the only possible case. Indeed, if

$$\frac{4(g-1) + K_S^2 + \sqrt{\Delta}}{2} \leq m,$$

then $C.K_S \leq 2(g-1) - m \leq 0$ that is impossible. \square

As a consequence of Lemma 3.2 we obtain our first general fact concerning the number s :

Proposition 3.3. *Let S be of general type. If $s = 6$, then*

$$K_S^2 + \sqrt{8(g-1)K_S^2 + (K_S^2)^2} \leq 6\chi(\mathcal{O}_S).$$

In particular, if $g \geq 6$ and $\chi(\mathcal{O}_S) = 1$, then $s \geq 7$.

Proof. Assume $s = 6$, by Tan's inequality:

$$\frac{1}{3}e(K_X^2e + 12(g-1)) \leq e_f,$$

for any natural number $e \geq 2$. Evaluating in $e = 3$ we obtain:

$$(2) \quad 3K_X^2 + 12(g-1) \leq e_f.$$

Since f is semistable $e(X) = -4(g-1) + e_f$ and (2) becomes:

$$(3) \quad 3K_X^2 + 12(g-1) \leq e_f = e(X) + 4(g-1).$$

By definition $m = K_S^2 - K_X^2 = e(X) - e(S)$. This, combined with Noether Formula $e(S) = 12\chi(\mathcal{O}_S) - K_S^2$ leads to:

$$K_S^2 + 2(g-1) \leq 3\chi(\mathcal{O}_S) + m.$$

The desired inequality follows after applying Lemma 3.2. \square

The importance of this Proposition is that given a family of general type surfaces with given invariants K_S^2 and $\chi(\mathcal{O}_S)$ only a finite numbers of values of g must be discharged in order to conclude that a fibered surface birational to S has at least 7 singular fibers. This principle will be illustrated in the next section.

Note that in Proposition 3.3 the hypothesis of being S of general type is essential ([2], Example 2).

4. FIBRATIONS OBTAINED FROM EXCEPTIONAL SURFACES

Surfaces satisfying either $K_S^2 = 2$ and $p_g = 3$ or $K_S^2 = 1$ and $p_g = 2$ are called exceptional because of the behavior of the tri-canonical map (Theorem VII 8.3 in [1]). In this section we study fibrations in these surfaces by means of the canonical and bi-canonical map, respectively.

Theorem 4.1. *Assume S satisfies $K_S^2 = 2$ and $p_g = 3$. Let $f : X \rightarrow \mathbb{P}^1$ be obtained as the blowing-up of a transversal pencil Λ in S . Then, $s \geq 7$.*

Proof. By Debarre's Inequality ([6]) we know that $q = 0$ and therefore $\chi(\mathcal{O}_S) = 4$. Assume $s = 6$, by Proposition 3.3 it is sufficient to consider:

$$6 \leq g \leq 31.$$

In this case the canonical map ϕ_{K_S} defines a $2 : 1$ covering:

$$\phi_{K_S} : S \rightarrow \mathbb{P}^2,$$

ramified along $R \equiv 4K_S$ ([8]). Consider the restriction $\phi := \phi_{K_S}|_C$.

First, we consider the case ϕ is a $2 : 1$ covering. Denote by $G \subset \mathbb{P}^2$ the image of C under ϕ and d for its degree. We have $C = \phi^*G$ and therefore:

$$m = C^2 = (\phi^*G)^2 = 2G^2 = 2d^2.$$

On the other hand, taking into account that $\phi^*H = K_S$, with H a hyperplane section (i.e. the divisor associated with $\mathcal{O}_{\mathbb{P}^2}(1)$) we obtain:

$$d = \frac{C.K_S}{2} \text{ and } m = \frac{(C.K_S)^2}{2}.$$

From this we get $2m = (2(g-1) - m)^2$. The possible values of g satisfying such a relation in the range $6 \leq g \leq 31$ are: $g = 7, 13, 21, 31$. The corresponding values of m and $C.K_S$ are listed below:

$g - 1$	$C.K_S$	m
6	4	8
12	6	18
20	8	32
30	10	50

If $g - 1 = 20$ or 30 we use the inequalities of Lemma 3.1 ii), in order to obtain a contradiction.

Assume $g - 1 = 6$. In this case the number of singular points in the fibers of f will be:

$$\begin{aligned} e_f &= e(X) + 4(g - 1) \\ &= -K_X^2 + 12\chi(\mathcal{O}_X) + 4(g - 1) \\ &= 6 + 48 + 24 = 78. \end{aligned}$$

Since $s = 6$ there exists at least a singular fiber F_0 of f containing $\sigma_0 = 13 = 78/6$ singular points. But then, denoting $F_0 = F_1 + \dots + F_l$ for the decomposition into irreducible components:

$$\begin{aligned}
6 = (g - 1) &= \sum_{i=1}^l (g_i - 1) + \sigma_0 \\
&\geq \sum (g_i - 1) + 13,
\end{aligned}$$

with g_i standing for the geometric genus of F_i and σ_0 for the number of singular points of F_0 . We have:

$$\sum_{i=1}^l (g_i - 1) \leq -7.$$

In particular, F_0 have at least 7 irreducible rational components. Being ϕ a $2 : 1$ covering and C be applied under ϕ to a plane curve G of degree $d = C.K_S/2 = 2$ we have that the number l of irreducible components of F_0 is at most 4. In this way we get a contradiction with the assumption $s = 6$.

The case $g - 1 = 12$ follows after similar considerations, this time tanking into account that the covering ϕ sends C onto a curve G of degree 3.

Consider now the case ϕ restricted to C is $1 : 1$. In this case there exists a curve C' such that

$$C + C' = \phi^*G.$$

Moreover, since the ramifications of ϕ over C occurs exactly on the intersections of C and C' we have $C.C' = C.R = C.4K_S$. Similarly, we conclude that $C.K_S = C'.K_S$. Also, we have that:

$$2C.K_S = \phi^*G.K_S = 2d,$$

i.e. $d = C.K_S$.

It follows that $\phi^*G = \phi^*((C.K_S)H) = (C.K_S)K_S$. From

$$C + C' = (C.K_S)K_S,$$

we obtain, after intersecting with C :

$$4K_S.C = C.C' = (C.K_S)^2 - m.$$

The possibilities for such a relationship are listed below:

$(g - 1)$	$C.K_S$	m
5	5	5
9	6	12
14	7	21
20	8	32
27	9	45

Using Lemma 3.1 we obtain a contradiction in the following cases: $g - 1 = 20$ and 27 when evaluating at $e = 4$, and $g - 1 = 5, 9$ when evaluating at $e = 5$.

Finally, we must analyze the case $g - 1 = 14$. In this case,

$$\begin{aligned}
e_f &= 4(g - 1) - K_X^2 + 12\chi(\mathcal{O}_X) \\
&= 123.
\end{aligned}$$

Assuming $s = 6$ there must to exist a singular fiber F_0 with its number of nodes $\sigma_0 \geq 21$. Assume $F_0 = F_1 + \dots + F_l$ is its decomposition into irreducible components. Denoting by g_i the geometric genus of F_i we have:

$$14 = \sum_{i=1}^l (g_i - 1) + \sigma_0 \geq \sum_{i=1}^l (g_i - 1) + 21.$$

From this we get $l \geq 7$. Now, the image of F_0 in \mathbb{P}^2 (under the composition $\phi \circ \pi$) is a degree 7 curve G_0 . It follows that $l = 7$, $G_0 = L_1 + \dots + L_7$ is the sum of seven lines L_i , and $C_0 := \pi(F_0)$ must be the sum of seven irreducible components $C_0 = C_1 + \dots + C_7$ ($C_i = \pi(F_i)$). We have, moreover, $\phi^* L_i = C_i + C'_i \equiv K_S$, $C_i^2 = C_i'^2 = -3$ and $C_i C'_i = 4$.

A simple cohomological computation shows that $h^0(\mathcal{O}_S(C_{i_1} + C_{i_2} + C_{i_3})) = 1$ for any indexes $i_1, i_2, i_3 \in \{1, \dots, 7\}$ and $h^0(\mathcal{O}_S(C_1 + \dots + C_4)) = 2$. This, together with $(C_1 + \dots + C_4)^2 = 0$ means that $|C_1 + \dots + C_4|$ is a base point free pencil. Call $\Delta := C_1 + \dots + C_4$. We have, $C'_i \cdot \Delta = 0$ for $i = 5, 6, 7$, thus $C'_5 + C'_6 + C'_7$, being connected, must be a vertical divisor with respect to $|\Delta|$. We conclude the existence of an effective divisor D' such that $\Delta \sim C'_5 + C'_6 + C'_7 + D'$. It is easy to deduce that D' is a rational (-3) -curve and $\phi(D')$ is a line in \mathbb{P}^2 .

From this relation and

$$(C_1 + C'_1) + \dots + (C_7 + C'_7) \sim 7K_S,$$

it follows that:

$$C \sim C_1 + \dots + C_7 \sim 3K_S + D'.$$

This gives a contradiction, as the image $\phi(C)$ is a degree 7 curve and the image $\phi(3K_S + D')$ is a degree 4 curve. \square

Theorem 4.2. *Assume $f : X \rightarrow \mathbb{P}^1$ is obtained as a blow up of a transversal pencil Λ on a minimal surface S with $K_S^2 = 1$ and $p_g = 2$. Then $s \geq 7$.*

Proof. By Proposition 3.3 we must consider only the values $6 \leq g \leq 37$.

Considering the classical Horikawa's construction and notation ([9]), let $\Pi : \bar{S} \rightarrow S$ be the blowing up centered in the base point p of $|K_S|$, denote by E its exceptional divisor and consider the ramified double covering:

$$\phi_2 : \bar{S} \rightarrow \mathbb{F}_2.$$

The map ϕ_2 is given as follow: the bicanonical map of S determines a double cover on the singular quadric $Q \subset \mathbb{P}^3$, the singular point being the image of p . ϕ_2 is the induced map on \bar{S} after considering the desingularization \mathbb{F}_2 of the quadric. The locus branch of ϕ_2 is the divisor $B = 6\Delta + 10\Gamma$, with Δ and Γ denoting respectively the class of the (-2) -section and the class of the fiber in \mathbb{F}_2 of the structural morphism and the ramification divisor R is $5K_S + E$. Here and in what follows given D any divisor in S we just write D for the divisor $\Pi^* D$ in \bar{S} .

Denote by $\bar{\Lambda}$ the induced pencil on \bar{S} . Depending on whether p is a base point of Λ or not we have

$$(4) \quad \bar{C} = \begin{cases} \Pi^*C & \text{if } p \notin \Lambda \\ \Pi^*C - E & \text{if } p \in \Lambda. \end{cases}$$

Let G be the image of \bar{C} under ϕ_2 . If we denote $G = a\Delta + b\Gamma$ and considering that $\phi^*\Delta = 2E$, $\phi^*\Gamma = K_S - E$ we have

$$(5) \quad \phi^*G = bK_S + (2a - b)E.$$

Let be $\phi := \phi_2|_{\bar{C}} : \bar{C} \rightarrow G$ with $\deg \phi = n = 1$ or 2 .

We analyze the two cases $n = 1$ or 2 and within each one the subcases $p \in \Lambda$ or not.

Case 1 ϕ is 2:1. In this case $\phi^*G = \bar{C}$.

Assume first that $p \notin \Lambda$.

By (4) and (5) we have that $2a - b = 0$, using this $\bar{C}^2 = m = b^2$. On the other hand $m = bC.K_S$, therefore $b = C.K_S$. The next table shows the possible values of m , $C.K_S$ and $g - 1$.

$C.K_S$	a	m	$g - 1$
4	2	16	10
6	3	36	21
8	4	64	36

Assume $g - 1 = 10$. In this case the number of singular points in the fibers of f will be:

$$\begin{aligned} e_f &= 12\chi(\mathcal{O}_X) - K_X^2 + 4(g - 1) \\ &= 36 + 15 + 4(10) = 91 \end{aligned}$$

Since $s = 6$ there exists at least a singular fiber F_0 of f containing 16 singular points. Let $F_0 = F_1 + \dots + F_l$ be the decomposition into irreducible components, then:

$$(6) \quad \begin{aligned} 10 = g - 1 &\geq \sum_{i=1}^l (g_i - 1) + \sigma_0 \\ &\geq \sum (g_i - 1) + 16 \end{aligned}$$

where g_i denotes the geometric genus of F_i . This imply that $l \geq 6$. Moreover there are at least 6 of this components that are rational curves and are mapping onto rational components of G_0 .

From [7] (Corollary V.5.18) we know that the possible irreducible curves in \mathbb{F}_2 are: Γ , Δ and $\alpha\Delta + \beta\Gamma$ with $\alpha > 0$, $\beta \geq 2a$. Let $G_0 = G_1 + \dots + G_s$ be the decomposition into irreducible components.

Note that even being Δ a rational curve, is not a possibility for any of the G_i 's, that because if $G_1 = \Delta$ then

$$\bar{C}_1 = \phi^*G_0 = 2E$$

that contradicts the semistability of f .

With respect to the components $G_i = a_i\Delta + b_i\Gamma$, $a_i > 0$, $b_i \geq 2a$, G_i is rational if and only if $a_i = 1$ and $b_i \geq 2$. Since $a = \sum_{i=1}^s a_i$ and $b = \sum_{i=1}^s b_i$ the only possible

decomposition is $G_0 = G_1 + G_2$ with $G_i = \Delta + 2\Gamma$, $i = 1, 2$, i.e. we can't have the 6 needed rational components.

If $g - 1 = 21, 36$ we use similar arguments with $\sigma_0 = 26, 41$ respectively. In both cases there exist at least 5 rational components and this is not possible because $a < 5$.

Now consider the case $p \in \Lambda$.

By (4) and (5) $b = 2a + 1$, so $m - 1 = \bar{C}^2 = 4a(a + 1)$ and $m = b^2$. Moreover,

$$\bar{C}.K_{\bar{S}} = C.K_S + 1 = b + 1$$

Therefore $m = (C.K_S)^2$. Keeping in mind the previous notation, we get the next possible values

$C.K_S$	a	m	$g - 1$	e_f	σ_0	$l \geq$
3	1	9	6	68	12	6
5	2	25	15	120	20	5
7	3	49	28	196	33	5

Observe that l is, as before, the minimal number of rational components in G_0 , so analogous to the case $p \notin \Lambda$, all possibilities in the table can't occur because of $a < l$.

Case 2 ϕ is 1:1. In this case there exists a divisor \bar{C}' such that $\phi^*G = \bar{C} + \bar{C}'$.

Denote as before by F_0 a singular fiber of f , C_0 its image under π . If $F_0 = F_1 + \dots + F_k$ is the decomposition of F_0 into irreducible components, we denote by $C_0 = C_1 + \dots + C_k$ the corresponding decomposition for C_0 and by $\bar{C}_0 = \bar{C}_1 + \dots + \bar{C}_k$ the corresponding curves and decomposition in \bar{S} and by $G_0 = G_1 + \dots + G_k$ their images in \mathbb{F}_2 .

We begin by stating the following:

Lemma 4.3. *In the previous situation, let G_1 be any irreducible component of G_0 , then neither $G_1 \sim \Delta$, nor $G_1 \sim \Gamma$ nor $G_1 \sim \Delta + 2\Gamma$. In particular, if $G_0 \sim a\Delta + b\Gamma$, then the number of rational irreducible components of G_0 is least or equal than a .*

Proof. Let G_1 be equivalent to Δ . Then $\phi_2^*(G_1) \sim 2E$. Note that there exists a divisor \bar{C}'_1 such that $\bar{C}_1 + \bar{C}'_1 = \phi_2^*(G_1)$. This implies $\bar{C}_1 = E$, which is impossible by the definition of \bar{C} .

Now, assume $G_1 \equiv \Gamma$, then $\phi^*G_1 \equiv K_S - E = C_1 + C'_1$. From this, intersecting with K_S , and using that K_S is nef we obtain that either $C_1.K_S = 0$ or $K_S.C'_1 = 0$. It follows that there exists a curve D on S with $K_S.D = 0$, which contradicts that ampleness of $|K_S|$.

Finally, suppose that $G_1 = \Delta + 2\Gamma$, in this case $\phi^*G_1 \equiv 2K_S$.

Note that for any decomposition $2K_S = A + A'$ we must have that both, A and A' must be irreducible and equivalent to K_S and $A.A' = 1$. Indeed, assume A irreducible, that from $2K_S = A + A'$ with easy it follows that $A.K_S = A'.K_S = 1$. Therefore, $A^2 = 1$, because $2K_S$ is 1-connected, $A \equiv K_S$ follows from HIT.

One established this fact, just note that if $2K_S = C_1 + C'_1$, then $\phi : C_1 \rightarrow G_1$ can not be 1 : 1, because C_1 is a genus 2 curve.

The last assertion follows from the fact that the only irreducible rational curves on \mathbb{F}_2 are equivalent to either Δ , Γ or $\Delta + b\Gamma$ with $b \geq 2$. \square

Continuing the proof of the Theorem, assume first that $p \notin \Lambda$. We have the commutative diagram:

$$\begin{array}{ccc} \bar{S} & \xrightarrow{\phi_2} & \mathbb{F}_2, \\ \Pi \downarrow & & \downarrow \\ S & \xrightarrow{\phi_{2K}} & Q \end{array}$$

therefore $\phi_{2K}(\Pi(\bar{C}'))$ contains the singular point of Q and from this it follows that $\bar{C}' = \Pi^*C'$, for some effective divisor C' in S . By (5) $b = 2a$ and $\bar{C} + \bar{C}' \sim bK_S$. Therefore,

$$C.K_S + C'.K_S = (\bar{C} + \bar{C}')K_{\bar{S}} = (bK_S)(K_S + E) = b.$$

The ramifications of ϕ_2 occurring on \bar{C} are given by the intersections of \bar{C} and \bar{C}' , so we have $\bar{C}.\bar{C}' = \bar{C}.R = \bar{C}'.R$. In particular, the right hand term of the previous equation implies that $C.K_S = C'.K_S$ and $b = 2C.K_S$.

Moreover,

$$bC.K_S = \phi^*G.\bar{C} = (\bar{C} + \bar{C}').\bar{C} = m + \bar{C}.\bar{C}' = m + 5C.K_S.$$

We conclude that $m = C.K_S(2C.K_S - 5)$ and we get the possible values (keeping in mind the previous notation introduced for σ_0 and l):

b	$C.K_S$	m	$g - 1$
8	4	12	8
10	5	25	15
12	6	42	24
14	7	63	35

The values $g - 1 = 24, 35$ are impossible because of the Hodge Index Theorem.

If $g - 1 = 8$ we have that $e_f = 79$ and then there must exist a singular fiber F_0 with at least 14 singular points. It follows that G_0 has 6 or more rational components. Using that $a = 4$ and Lemma 4.3, we get a contradiction. The case $g - 1 = 15$ follows after similar considerations.

It remains to analyze the case $p \in \Lambda$.

As in the previous case, like $\bar{C} = C - E$ also $\bar{C}' = C' - E$. From this we get $2a - b = -2$. Therefore we have:

$$\phi_2^*G = C + C' - 2E = bK_S - 2E.$$

Moreover, $\bar{C}.R = \bar{C}'.R$ and then $C.K_S + 1 = \bar{C}.K_{\bar{S}} = \bar{C}'.K_{\bar{S}}$

We get the next formulas

$$\phi_2^*G.K_{\bar{S}} = b + 2 = 2C.K_S + 2$$

and therefore

$$m + 5C.K_S = \phi_2^*G.\bar{C} = (bK_S - 2E).(C - E) = bC.K_S - 2$$

We conclude that $b = 2C.K_S$ and $m = 2(C.K_S)^2 - 5C.K_S - 2$. The table of possible values is:

$C.K_S$	b	a	m	$g-1$
4	8	3	10	7
5	10	4	23	14
6	12	5	40	23
7	14	6	61	34

If $g-1 = 23, 34$ we get a contradiction by Hodge Index Theorem. If $g-1 = 7$, then $e_f = 73$, therefore there exists a singular fiber F_0 with at least 13 singular points and at least 6 rational components. Taking in consideration that $a = 3$ and Lemma 4.3 we obtain a contradiction. The case $g-1 = 14$ is similar. \square

5. THE ADJOINT CASE

In this section we consider fibrations $f : X \rightarrow \mathbb{P}^1$ satisfying the property that C is an adjoint linear system, i.e., $C \equiv B + K_S$ with B a big and nef divisor. The typical example for bearing in mind is $C \equiv nK_S$, i.e. the fibration f is obtained after blowing up the base locus of a generic pencil of curves $\Lambda \subset |nK_S|$.

We collect, for further use, some general elemental facts in the following:

Lemma 5.1. *Assume $C \equiv B + K_S$ with B a big and nef divisor, denoting by g_B the arithmetic genus of B , we have:*

i)

$$2(g_B - 1) = B^2 + B.K_S.$$

ii)

$$m = (g_B - 1) + (g - 1).$$

iii)

$$(g - 1) = (g_B - 1) + B.K_S + K_S^2.$$

iv) $2 \leq B.K_S$, and if $g \geq 5$, then

$$g + 1 \leq m.$$

Proof. Assertions i)-iii) follow immediately from adjunction formula. As for iv), note that it is enough to prove that $2 \leq B.K_S$, because then using i) $g_B - 1 \geq 2$ and the desired inequality follows from ii). Now, if $B.K_S = 1$, by Index Hodge Theorem $B^2 = K_S^2 = 1$ and applying from $m = B^2 + 2B.K_S + K_S^2$ we have $m = 4$. On the other hand, from i) and ii) $m = 1 + (g - 1) = g \geq 5$. \square

We start by studding the case $K_S^2 = 1$, which is similar in nature to Theorems 4.1 and 4.2:

Proposition 5.2. *Let $f : X \rightarrow \mathbb{P}^1$ be a fibration obtained as the blowing up of the base locus of a transversal and adjoint pencil Λ on a minimal surface with $K_S^2 = 1$. Then, $s \geq 7$.*

Proof. By Noether Inequality $p_g \leq 2$, and by Proposition 3.3 and Theorem 4.2, we can assume $p_g = 1$. It is well known that for a surface with such invariants the bicanonical map ϕ_{2K_S} defines a 4 : 1 morphism onto \mathbb{P}^2 ([5], [10], [14]):

$$\phi_{2K_S} : S \rightarrow \mathbb{P}^2,$$

ramified along a divisor $R \equiv 7K_S$. We consider, as before, the restriction of this map to C :

$$\begin{array}{ccc}
C & \xrightarrow{n:1} & \tilde{G} \\
\phi \downarrow & \swarrow j & \\
G & &
\end{array}$$

In this case n is a divisor of 4, $G \subset \mathbb{P}^2$ is the image of C and j denotes its normalization. Denote by d the degree of G .

If we assume $s = 6$ we only need, according to Proposition 3.3 to consider $6 \leq g \leq 16$.

We start by analyzing the case $n = 4$: in this case we have $C = \phi^*G$ and therefore:

$$m = C^2 = (\phi^*G)^2 = 4G^2 = 4d^2.$$

On the other hand, taking in account that $\phi^*H \equiv 2K_S$, with H hyperplane section (i.e the divisor associated with $\mathcal{O}_{\mathbb{P}^2}(1)$) we obtain:

$$C.K_S = \phi^*G.K_S = \phi^*(dH).K_S = 2d.$$

Adjunction formula gives $2(g-1) = 2d(2d+1)$. The only value of g that satisfies the relation in the range $6 \leq g \leq 16$ is $g-1 = 10$ with $d = 2$ and $m = 16$. Evaluating in Tan's inequality for $e = 4$ (Lemma 3.1 ii)) we obtain a contradiction.

If $n < 4$, then there exists an effective divisor $C' > 0$ such that:

$$C + C' \equiv \phi_{2K_S}^*G \equiv 2dK_S.$$

Note that:

$$C'.K_S = 2d - C.K_S,$$

and

$$C'^2 = 4d^2 - 4dC.K_S + C^2.$$

Moreover, since $h^1(C) = h^2(C) = 0$,

$$h^0(C) = \frac{C^2 - C.K_S}{2} + 2.$$

Next, the only possibility for $h^2(C') = h^0(K_S - C') \neq 0$ is $C' \equiv K_S$, because $p_g = 1$. This would imply $C \equiv K_S$, which is impossible.

It is easy to prove that $H^0(C) \simeq H^0(C')$ and by Riemann-Roch we get:

$$h^0(C) = h^0(C') \geq \frac{C'^2 - C'.K_S}{2} + 2,$$

and substituting the values of C'^2 and $C'.K_S$:

$$h^0(C) \geq h^0(C') + \frac{4d^2 - (4d-2)C.K_S - 2d}{2}.$$

This implies,

$$2d^2 - (2d-1)C.K_S - d \leq 0,$$

that is equivalent to $d \leq C.K_S$.

Now, assume $n = 1$. The linear system $|2K_S| |_C$ defines a base point free linear system on C and the associated map a $1 : 1$ cover onto a plane degree d curve.

Thus, we have $d = 2C.K_S$ and we obtain a contradiction with the just obtained bound $d \leq C.K_S$.

The case $n = 2$ remains to be analyzed: in this case we have $C.K_S = d$. Note that the intersections of C and C' gives place to ramifications points of ϕ_{2K_S} . Therefore:

$$2dC.K_S - m = C.C' \leq C.R = 7C.K_S.$$

From this we get:

$$d(2d - 7) \leq m \leq 2d^2 - 2.$$

It follows that $d \leq 5$. In general we have that

$$\begin{aligned} e_f &= e(X) + 4(g - 1) \\ (7) \quad &= 24 - (1 - m) + 4(g - 1), \\ &= 23 - d + 6(g - 1). \end{aligned}$$

Thus, assuming $s = 6$ and $d \leq 3$, there must exists a singular fiber F_0 of f having at least $(g - 1) + 4$ nodes. Call σ_0 the number of nodes of F_0 . Note that the number of nodes of $C_0 = \pi(F_0)$ is also σ_0 . Then, $\sigma_0 \geq 9$. On the other hand, the plane curve $G_0 = \phi_{2K_S}(C_0)$, being of degree $d \leq 3$ admits at most 3 nodes. In this way we get the contradiction $\sigma_0 \leq 6$.

Similar argumentations lead to contradictions for the cases $d = 4, 5$. Indeed, if $d = 4$, then $\sigma_0 \geq g + 3$. We have:

$$g - 1 = \sum_{i=1}^l (g_i - 1) + \sigma_0 \geq -l + g + 3,$$

with g_i standing for the geometric genus of the components F_i of F_0 . It follows that $l \geq 4$ and therefore F_0 (and in consequence C_0) has at least 4 rational components. From this it follows that G_0 has at least 2 irreducible rational components. Taking in account that G_0 is a degree 4 curve we have that, either G_0 contains a line as an irreducible component or it is the product of two irreducible conics. If G_0 is the product of two irreducible conics then it has only 4 nodes and we get, as before a contradiction, in any other case, if L is an irreducible component of G_0 and $C_0 = C_1 + \dots + C_l$, then

$$\phi_{2K_S}^* L = C_i + C'_i,$$

for C_i some rational components of C_0 . But then:

$$2K_S \equiv C_i + C'_i,$$

and it follows ([4] Lemma 1, page 181) that $C_i = \Delta$, the only effective divisor in $|K_S|$. This give a contradiction, since Δ is a curve of geometric genus 2.

Finally the case $d = 5$ follows after similar considerations. In this case C_0 admits at least 3 irreducible rational components and G_0 at least 2 irreducible rational components. The only subtle case to be treated careful being the possibility that $G_0 = Q + E$, with Q an irreducible conic and E a singular irreducible cubic. But in this case either Q or E must satisfies that it pull back under ϕ_{2K_S} is the sum of two irreducible components $C_i + C_j$ of C_0 .

Note that $\phi_{2K_S}^* E = C_i + C_j$ is impossible, because then

$$6K_S \equiv C_i + C_j$$

and then $K_S.(C_i + C_j) = 6$, that contradicts $K_S.C_0 = 5$. On the other hand

$$4K_S = \phi_{2K_S}^* Q = C_i + C_j$$

implies that C_0 has exactly 3 irreducible components: $C_0 = C_1 + C_2 + C_3$. Suppose $i = 1$ and $j = 2$, then $C_3.K_S = 1$ and

$$\phi_{2K_S} : C_3 \rightarrow E,$$

must be a $2 : 1$ map onto a degree cubic and we obtain the contradiction

$$2K_S.C_3 = 6.$$

□

Finally we have:

Theorem 5.3. *Let $f : X \rightarrow \mathbb{P}^1$ a semi-stable non-isotrivial fibration obtained as the blow-up of the base locus of an adjoint pencil Λ on the minimal surface S . Then if $K_S^2 \geq 3$ the number s of singular fibers of f is at least 7. If $K_S^2 \leq 2$ and Λ is also transversal, then $s \geq 7$.*

Proof. We assume $s = 6$ and $2 \leq K_S^2$. From Noether's inequality:

$$p_g \leq \frac{K_S^2}{2} + 2,$$

we have:

$$\chi(\mathcal{O}_S) \leq \frac{K_S^2}{2} + 3.$$

Applying Lemma 3.3:

$$K_S^2 + \sqrt{8K_S^2(g-1) + (K_S^2)^2} \leq 3K_S^2 + 18,$$

which implies

$$8K_S^2(g-1) + (K_S^2)^2 \leq 4(K_S^2)^2 + 72K_S^2 + 18^2.$$

Substituting $2(g-1) \geq C.K_S + m = B.K_S + K_S^2 + m$, we get:

$$m \leq \frac{72 - 4B.K_S}{4} + \frac{18^2 - (K_S^2)^2}{4K_S^2}.$$

Now, combine the previous bound for m with Lemma 3.1 ii), in order to deduce:

$$19K_S^2 + 19B.K_S \leq 18 + \frac{18^2 - (K_S^2)^2}{4K_S^2} + 108.$$

Using again the adjoint hypothesis, this amount to:

$$\frac{(K_S^2)}{4} + 19C.K_S \leq 166,$$

that is,

$$C.K_S \leq 8.$$

Now, use Hodge Index Theorem:

$$m \leq \frac{(C.K_S)^2}{K_S^2} \leq \frac{64}{K_S^2},$$

and apply one more time Lemma 3.1 ii):

$$K_S^2 + 18C.K_S \leq \frac{64}{K_S^2} + 108,$$

$$19K_S^2 + 18B.K_S \leq \frac{64}{K_S^2} + 108.$$

Finally, using $2 \leq B.K_S$, (Lemma 5.1 iv)) we arrive to:

$$19K_S^2 \leq \frac{64}{K_S^2} + 72.$$

This implies $K_S^2 \leq 4$. Thus, only the cases $K_S^2 = 2, 3, 4$ remains to be discharged. This is easy and essentially is a reproduction of the previous argument.

For instance, for case $K_S^2 = 2$ we have by Lemma 3.2 and Proposition 3.3 that $g \leq 16$ and $m \leq 26$, moreover, assuming Λ is transversal, we can apply Theorem 4.1 and assume that $\chi(\mathcal{O}_S) \leq 3$. Evaluating Tan's Inequality at $e = 5$ (Lemma 3.1 iii)) we get:

$$7K_S^2 + 6C.K_S \leq m + 9\chi(\mathcal{O}_S),$$

that, under our fixed values becomes $C.K_S \leq 6$. Using Hodge Index Theorem we obtain $m \leq 18$. Evaluating again Tan's Inequality at $e = 4$ we have $C.K_S \leq 4$ and $m \leq 8$. Finally, we evaluate once again Tan's Inequality at $e = 5$ and get the final contradiction $C.K_S \leq 3$ and $g + 1 \leq m \leq 4$.

Cases $K_S^2 = 3, 4$ are quite analogous, only that in these cases we don't need Theorem 4.1, and in consequence the transversality hypothesis can be avoided. \square

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